Dynamic Systems
Sea-Level Rise Model:
Interpretation of the Forcing Function

Jiabao Guan, Biao Chang and Mustafa M. Aral
Multimedia Environmental Simulations Laboratory
School of Civil and Environmental Engineering
Georgia Institute of Technology
Climate Change / Sea-level Rise
DATA on:

Temperature
Sea-Level Rise
Radiative forces (CO$_2$ Emissions)
Global Temperature Change:
Global Sea-Level Rise:
Data on CO$_2$ Emissions:
The **Summary for Policy Makers** (SPM) released recently provide the following table of sea level rise projections (IPCC4th Framework Report, 2007):

<table>
<thead>
<tr>
<th>Case</th>
<th>Sea Level Rise (m at 2090-2099 relative to 1980-1999)</th>
<th>Model-based range excluding future rapid dynamical changes in ice flow</th>
</tr>
</thead>
<tbody>
<tr>
<td>B1 scenario</td>
<td>0.18 - 0.38</td>
<td></td>
</tr>
<tr>
<td>A1T scenario</td>
<td>0.20 - 0.45</td>
<td></td>
</tr>
<tr>
<td>B2 scenario</td>
<td>0.20 - 0.43</td>
<td></td>
</tr>
<tr>
<td>A1B scenario</td>
<td>0.21 - 0.48</td>
<td></td>
</tr>
<tr>
<td>A2 scenario</td>
<td>0.23 - 0.51</td>
<td></td>
</tr>
<tr>
<td>A1FI scenario</td>
<td>0.26 - 0.59</td>
<td></td>
</tr>
</tbody>
</table>
IPCC estimates:
Semi-Empirical Models:

Rahmstorf’s Study (Science, Vol. 315 pp.19, 2007) and others:
Semi-Empirical Models:

Rahmstorf’s Study (Science, Vol. 315 pp.19, 2007) and others.

Sea-level Rise Interval Predicted above 1990 level:

\[0.5 - 1.4 \text{ m}\]

IPCC Interval Predicted above 1990 level:

\[0.09 - 0.88 \text{ m}\]
Major limitations of the previous empirical models

- No feedback of SLR on temperature change
- Zero-dimensional, thus does not capture spatial variations in SLR
- Impact of external forcing is not considered
A Dynamic Systems Model
Earlier studies showed that the relationship between $T$ & $H$ is linear.

Our Hypothesis: *(Dynamic Systems Model)*:

- **Temperature:** $T = f_1(T, H, U, c_1)$
- **Sea-Level:** $H = f_2(T, H, U, c_2)$
Initial Model Proposed:  
(Simplified)

\[
\frac{dT(t)}{dt} = a_{11}T(t) + a_{12}H(t) + c_1 \\
\frac{dH(t)}{dt} = a_{21}T(t) + a_{22}H(t) + c_2
\]
Initial Model Proposed:  
(Simplified)

\[
\frac{dX(t)}{dt} = \left( \frac{dT(t)}{dt}, \frac{dH(t)}{dt} \right) \tau
\]

\[
\frac{dX(t)}{dt} = AX(t) + C
\]

\[
A = \begin{bmatrix}
    a_{11} & a_{12} \\
    a_{21} & a_{22}
\end{bmatrix}
\]

\[
C = \begin{bmatrix}
    c_1 \\
    c_2
\end{bmatrix}
\]
Initial Model Proposed:
(Simplified)

A Dynamic System Model to Predict Global Sea-Level Rise and Temperature Change

Mustafa M. Aral*, Jiabao Guan and Biao Chang
Dynamic Systems Model:

\[ U(t) \]

\[ X(t) \]

\[ \frac{dX}{dt} \]

\[ X(t) \rightarrow \text{System} \rightarrow \frac{dX}{dt} \]
Proposed Model:

\[
\frac{dT(t)}{dt} = a_{11} T(t) + a_{12} H(t) + \sum_{i} a_{13i} U_i(t) + c_1
\]

\[
\frac{dH(t)}{dt} = a_{21} T(t) + a_{22} H(t) + \sum_{i} a_{23i} U_i(t) + c_2
\]
The Proposed model is:

- more flexible;
- may answer more questions;
- may provide control analysis perspective; and,
- hopefully will be more useful.
- potential drawback may require more data.
Proposed Model:

\[
\frac{dX(t)}{dt} = AX(t) + BU(t) + C + w(t)
\]

\[
A = \begin{bmatrix}
a_{11} & a_{12} \\
a_{21} & a_{22}
\end{bmatrix}
\]

\[
B = \begin{bmatrix}
b_{11} & b_{12} & \cdots & b_{1m} \\
b_{21} & b_{22} & \cdots & b_{2m}
\end{bmatrix}
\]

\[
C = \begin{bmatrix}
c_1 \\
c_2
\end{bmatrix}
\]
Discrete form of the Proposed Model:

\[ X(\Phi X_1) = \Gamma U(k) + \Omega (k) + + 1(k) \]

\[ \Phi = I + A\Delta t ; \quad \Gamma = B\Delta t ; \quad \Omega = C\Delta t \]
LSM Model to determine $a_{ij}, b_{ij}$ & $c_i$:

$$F^* = \minimize_{\phi_i} \left\{ \left( Y_i \Lambda \phi_i \right) \tilde{Y} \left( \Lambda \phi_i \right) \right\}$$
Confidence Interval:

\[ \hat{T}_{CI}(k) = \hat{T}(k) \pm t_{\alpha/2, n-4} \sqrt{\hat{\sigma}^2_T \left(1 + \hat{Z}(k)^\tau \left(\Lambda^\tau \Lambda\right)^{-1} \hat{Z}(k)\right)} \]

\[ \hat{H}_{CI}(k) = \hat{H}(k) \pm t_{\alpha/2, n-4} \sqrt{\hat{\sigma}^2_H \left(1 + \hat{Z}(k)^\tau \left(\Lambda^\tau \Lambda\right)^{-1} \hat{Z}(k)\right)} \]

(Wadsworth, 1998)
Application:

- 2-year moving average outcome is used for both state variables.
Temperature Data:
Sea-Level Data:

Graph showing sea-level changes over time, with original data and moving averages.
### Resulting Matrix Coefficients:

<table>
<thead>
<tr>
<th>System</th>
<th>System matrix</th>
<th>Control matrix</th>
<th>Constant vector</th>
</tr>
</thead>
<tbody>
<tr>
<td>Discrete</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td>$\begin{bmatrix} 0.75042 &amp; -0.00053 \ 0.41015 &amp; 0.99568 \end{bmatrix}$</td>
<td>$\begin{bmatrix} 0.00245 \ 0 \end{bmatrix}$</td>
<td>$\begin{cases} -0.85744 \ 0.25863 \end{cases}$</td>
</tr>
<tr>
<td>Continuous</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td>$\begin{bmatrix} -0.24958 &amp; -0.00053 \ 0.41015 &amp; -0.00432 \end{bmatrix}$</td>
<td>$\begin{bmatrix} 0.00245 \ 0 \end{bmatrix}$</td>
<td>$\begin{cases} -0.85744 \ 0.25863 \end{cases}$</td>
</tr>
</tbody>
</table>
Global CO₂ Impact: Calibration

R² = 0.80

R² = 0.97
Global CO₂ Impact: Prediction for “2 °C” Scenario

(Scenario is designed to limit global warming in 2100 2°C above the temperature in 2000)

(Hansen et al., 2000)
IPCC Global CO₂ Emission Scenarios:
Global CO\textsubscript{2} Impact: Comparison of results

Solid lines are Dyn. Sys. Model results.

Dashed lines are IPCC results.
For example: If we want to restrict the global temperature and sea-level rise to zero-growth, the global CO₂ emission should be controlled with the relationship given by

\[ u_{CO_2}(t) = 101.87T(t) + 0.22H(t) + 349.98 \]

where

\[ u_{CO_2}(t) \] represents the amount of yearly CO₂ emission (ppm).
MESL @ GT

maral@ce.gatech.edu